RELATIVISTIC QUANTUM MECHANICS

Wednesday 06-05-2015, 18.30-21.30

Write your name and student number on **all** sheets. The total number of points is 90. Use conventions with $\hbar = c = 1$.

PROBLEM 1: LADDER OPERATORS (6+6+6=18 points)

1.1 Give the (anti-)commutation relations between $a_{\vec{p}}$ and $a_{\vec{q}}$, between $a_{\vec{p}}$ and $a^\dagger_{\vec{q}}$, and between $a^\dagger_{\vec{p}}$ and $a^\dagger_{\vec{q}}$ (all of which are bosonic operators).

1.2 Give the (anti-)commutation relations between $b_{\vec{p}}$ and $b_{\vec{q}}$, between $b_{\vec{p}}$ and $b^\dagger_{\vec{q}}$, and between $b^\dagger_{\vec{p}}$ and $b^\dagger_{\vec{q}}$ (all of which are fermionic operators).

1.3 Give the (anti-)commutation relations between $a_{\vec{p}}$ and $b_{\vec{q}}$, between $a_{\vec{p}}$ and $b^\dagger_{\vec{q}}$, and between $a^\dagger_{\vec{p}}$ and $b^\dagger_{\vec{q}}$.

PROBLEM 2: HAMILTONIAN FORMALISM (6+6+6=18 points)

The Lagrangian for a relativistic massive spinor field ψ is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \,. \tag{1}$$

2.1 What is the Euler-Lagrange equation for ψ?

2.2 What is the momentum conjugate to ψ, and what is the corresponding Hamiltonian density?

2.3 What are the Hamiltonian equations for this theory? Indicate the relations to the result of question 2.1.

PROBLEM 3: CAUSALITY (6+6+6=18 points)

3.1 Explain the notion of causality in a classical theory in simple terms.

3.2 Does causality imply that the product of two quantum operators that are separated via a space-like trajectory vanishes? Does the Compton wavelength play any role here?

3.3 Does causality imply that the (anti-)commutator of two quantum operators that are separated via a space-like trajectory vanishes? Does the Compton wavelength play any role here?

PROBLEM 4: HAMILTIONIAN OPERATOR (6+18+6+6=36 points)

The Hamiltonian operator for a relativistic massive scalar field ϕ is given by

$$H = \int d^3x \, (\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2) \, .$$

4.1 What is the expression for the field ϕ and its conjugate momentum π in terms of ladder operators $a_{\vec{p}}$ and $a_{\vec{p}}^{\dagger}$?

4.2 Derive the expression for the Hamiltonian operator in terms of ladder operators. Interpret the physical significance of your resulting expression.

4.3 Calculate the action of H on $a_{\vec{p}}^{\dagger}|0>$, where $|0>$ is the groundstate of H with eigenvalue 0.

4.4 Calculate the action of H on $a_{\vec{p}}^{\dagger}|E_0>$, where $|E_0>$ is an eigenstate of H with eigenvalue E_0.